Exploring the Geometry of Spacetime via Geodesic Curves

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**Kaustav Sen, Arastu Pandey,
Ira Sharma, Prajakta Mane**

*Indian Institute of Science Education and Research
Mohali
India*

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Abstract

Humans are funny creatures. They live in a 4-dimensional universe, yet solve problems of 3 dimensions. This term paper aims to drive into the reader's mind how something as simple as walking from your bed to your bathroom is actually a complex path called geodesic in 4D space. But let's not drive you in circles, and get straight to the point.

Introduction

Our intuition tells us that every square should close. The world is far stranger than our intuition would have us believe.The geometry does not have to work as per our intuition. When it does, it is called Euclidean. But in the vast majority of cases when it does not, it is called non-Euclidean geometry.

Each Non-Euclidean geometry is a consistent system of definitions, assumptions, and proofs that describe such objects as points, lines, and planes. The two most common non-Euclidean geometries are spherical geometry and hyperbolic geometry. The property of spherical geometry is that the sum of the angles of a triangle is always greater than 180° whereas it's less than 180° in hyperbolic.

As an example of this, take a spherical ball and imagine yourself to be an ant living on the ball (yep, that's how you imagine stuff). Now start traveling from any point on the equator of the ball and head for either of its poles to cover a certain distance. From the pole, turn by 90° and cover the same distance as was covered previously. You'll reach somewhere on the equator again. Again, turn to the same direction as before by 90° and walk the same distance again. Make the same turn again and you'll be back where you started with in the same orientation as the initial. Hence, it takes you three right-angle turns to return to the initial point and the orientation in this case. Also, if we measure the angles of this triangle covered by the ant, they will not add up to 180° either.

Now, imagine yourself to be the same ant on the same ball and traveling again (ants in maths always travel). But now the difference is you're not traveling till the pole. Start from a point, and take 5 steps (ant steps, to be precise), then keep turning by 90° and traveling the same distance till you reach the initial point with initial orientation like in the previous example. How many turns did you need to take? Intuitively, shouldn't it be 3, as it was in the previous example?

The number of turns would have been 4, tracing a square. This difference arises as a result of the difference between Euclidean and non-Euclidean

Geometries, as mentioned above. On the small scale, the world would look perfectly Euclidean to you. Returning to the initial point of the journey would take 4 right-angle turns. The same is not the case with the non-Euclidean space though.

A geodesic is a curve on a surface that a person on the surface would perceive as straight, while to a person capable of viewing from a point not on the surface, the path may have non zero curvature. Lets us take the earth for example. A "straight" highway from Delhi to Bangalore would obviously have to be curved, if it is to stay on this planet.[1]

If the surface does have non zero curvature, in order to stay on the path, the tangential component of acceleration must be 0. If it wasn't, we would simply fly off the surface (assuming the surface is smooth). Or equivalently, acceleration of an object on a curve on a surface can only have a component in a direction perpendicular to direction of motion. Hence acceleration has to be perpendicular to tangent.

A curve γ on a surface S is called a geodesic if $\gamma(t)$ is zero or perpendicular to the tangent plane of the surface at the point $\gamma(t)$, i.e., parallel to its unit normal, for all values of the parameter t.

We now make the claim that any geodesic has constant speed.

Proof. Let γ(t) be a geodesic on a surface S. Then, denoting d/dt by a dot,

$$\frac{d}{dt} \parallel \dot{\gamma} \parallel^2 = \frac{d}{dt}(\dot{\gamma} \cdot \dot{\gamma}) = 2\ddot{\gamma} \cdot \dot{\gamma}$$

Since γ is a geodesic, $\ddot{\gamma}$ is perpendicular to the tangent plane and is therefore perpendicular to the tangent vector $\dot{\gamma}$. So $\ddot{\gamma} \cdot \dot{\gamma} = 0$ and the last equation shows that $\parallel \dot{\gamma} \parallel$ is constant.

□

Geodesics preserve a direction on a surface (Tietze 1965, pp. 26-27) and have many other interesting properties. The normal vector to any point of a geodesic arc lies along the normal to a surface at that point (Weinstock 1974, p. 65).

Furthermore, no matter how badly a sphere is distorted, there exist an infinite number of closed geodesics on it. This general result, demonstrated in the early 1990s, extended earlier work by Birkhoff, who proved in 1917 that there exists at least one closed geodesic on a distorted sphere, and Lyusternik

[1] This argument assumes that the earth is an ellipsoid. The proof is left as an exercise to the reader.

and Schnirelmann, who proved in 1923 that there exist at least three closed geodesics on such a sphere

Suppose $f_S(t)$ is a function that gives a path between two points on a surface S, based on the parameter t. Minimizing $f_S(t)$ would hence give us the shortest path, which is a geodesic. A geodesic is therefore a locally length-minimizing curve. Equivalently, it is a path that a particle which is not accelerating would follow. In the plane, the geodesics are straight lines. On the sphere, the geodesics are great circles (like the equator). The geodesics in a space depend on the Riemannian metric, which affects the notions of distance and acceleration.

We now attempt to calculate the equation of a geodesic on an arbitrary surface. We use the help of the following equation, more commonly known as the Euler-Lagrange Equation[2] :

$$\boxed{\frac{\partial L}{\partial v} - \frac{d}{du}\left(\frac{\partial L}{\partial v'}\right) = 0} \tag{1}$$

For a surface given parametrically by $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$, the geodesic can be found by minimizing the arc length

$$I \equiv \int ds = \int \sqrt{dx^2 + dy^2 + dz^2} \tag{2}$$

But

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \tag{3}$$

$$dx^2 = \frac{\partial x}{\partial u}^2 du^2 + 2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} du dv + \frac{\partial x}{\partial v}^2 dv^2 \tag{4}$$

and similarly for dy^2 and dz^2. Plugging in

[2]Euler-Lagrange equation is a second-order partial differential equation whose solutions are the functions for which a given functional is stationary. Because a differentiable functional is stationary at its local extrema, the Euler–Lagrange equation is useful for solving problems in which, given some functional, one seeks the function minimizing or maximizing it.

$$I \equiv \int \left\{ \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \right] du^2 + 2 \left[\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \right] du dv + \left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \right] dv^2 \right\}^{1/2} \quad (5)$$

This can be rewritten as

$$I = \int \sqrt{P + 2Qv' + Rv'^2}\ du \quad (6)$$

$$= \int \sqrt{Pu'^2 + 2Qu' + R}\ dv \quad (7)$$

where

$$v' \equiv \frac{\mathrm{d}v}{\mathrm{d}u} \quad (8)$$

$$u' \equiv \frac{\mathrm{d}u}{\mathrm{d}v} \quad (9)$$

and

$$P \equiv \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \quad (10)$$

$$Q \equiv \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \quad (11)$$

$$R \equiv \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \quad (12)$$

Starting with equation (5)

$$I = \int \sqrt{P + 2Qv' + Rv'^2}\ du \quad (13)$$

$$= \int L du \quad (14)$$

and taking derivatives,

$$\frac{\partial L}{\partial v} = \frac{1}{2\sqrt{P + 2Qv' + Rv'^2}} \left(\frac{\partial P}{\partial v} + 2\frac{\partial Q}{\partial v} v' + \frac{\partial^2 R}{\partial v \partial v'}{}^2 \right) \quad (15)$$

$$\frac{\partial L}{\partial v'} = \frac{1}{2\sqrt{P + 2Qv' + Rv'^2}}\ (2Q + 2Rv') \tag{16}$$

so the Euler-Lagrange differential equation then gives

$$\frac{\frac{\partial P}{\partial v} + 2\frac{\partial Q}{\partial v}v' + \frac{\partial R}{\partial v}v'^2}{2\sqrt{P + 2Qv' + Rv'}} - \frac{d}{du}\left(\frac{Q + Rv'}{\sqrt{P + 2Qv' + Rv'^2}}\right) = 0 \tag{17}$$

Solving the above differential is a herculean task for most surfaces. However, natural surfaces are often symmetric, and we can use this symmetry to our advantage.[3]

In the special case when P, Q, and R are explicit functions of u only,

$$\frac{Q + Rv'}{\sqrt{P + 2Qv' + Rv'^2}} = c_1 \tag{18}$$

$$\frac{Q^2 + 2QRv' + R^2v'^2}{P + 2Qv' + Rv'^2} = {c_1}^2 \tag{19}$$

$$v'^2R(R - {c_1}^2) + 2v'Q(R - {c_1}^2) + (Q^2 - P{c_1}^2) = 0 \tag{20}$$

$$v' = \frac{1}{2R(R - {c_1}^2)}[2Q({c_1}^2 - R) \pm \sqrt{4Q^2(R - {c_1}^2)^2 - 4R(R - {c_1}^2)(Q^2 - P{c_1}^2)}] \tag{21}$$

Now, if P and R are explicit functions of u only and Q=0[4]**,**

$$v' = \frac{\sqrt{4R(R - {c_1}^2)P{c_1}^2}}{2R(R - {c_1}^2)} = c_1\sqrt{\frac{P}{R(R - {c_1}^2)}} \tag{22}$$

The equation of a geodesic on such a surface is then,

$$\boxed{v = c_1 \int \sqrt{\frac{P}{R(R - {c_1}^2)}}du} \tag{23}$$

In the case Q=0 where P and R are explicit functions of v only, then

[3] Such equations are known as Clairaut's relation. It is the main classical tool used to get qualitative information about geodesics on surface of revolution.

[4] The constant C_1 is called the Clairaut's constant.

$$\frac{\frac{\partial P}{\partial v} + v'^2 \frac{\partial R}{\partial v}}{2\sqrt{P + Rv'^2} - \frac{d}{du}\frac{Rv'}{\sqrt{P+Rv'^2}}} = 0 \tag{24}$$

So,

$$\frac{\partial P}{\partial v} + v'^2 \frac{\partial R}{\partial v} - 2\sqrt{P + Rv'^2}\; R\; \left[\frac{v''}{\sqrt{P + Rv'^2}} + (-\frac{1}{2})\frac{v'(2Rv'v'')}{(P + Rv'^2)^{3/2}}\right] = 0 \tag{25}$$

$$\frac{\partial P}{\partial v} + v'^2 \frac{\partial R}{\partial v} - 2Rv'' + \frac{2Rv'^2 v''}{P + Rv'^2} = 0 \tag{26}$$

$$\frac{Rv'^2}{\sqrt{P + Rv'^2}} - \sqrt{P + Rv'^2} = c_1 \tag{27}$$

$$Rv'^2 - (P + Rv'^2) = c_1\sqrt{P + Rv'^2} \tag{28}$$

$$\left(-\frac{P}{c_1}\right)^2 = P + Rv'^2 \tag{29}$$

$$\frac{P^2 - {c_1}^2 P}{R{c_1}^2} = v'^2 \tag{30}$$

,

The equation of a geodesic on such a surface is then,

$$\boxed{u = c_1 \int \sqrt{\frac{R}{P^2 - {c_1}^2 P}}\; dv} \tag{31}$$

If a surface of revolution in which y=g(x) is rotated about the x-axis so that the equation of the surface is

$$y^2 + z^2 = g^2(x), \tag{32}$$

the surface can be parameterized by

$$x = u \tag{33}$$

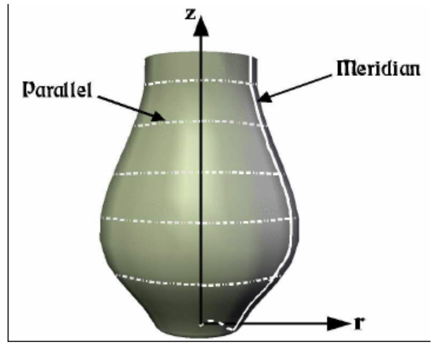
$$y = g(u) cos\ v \tag{34}$$

$$z = g(u) sin\ v \tag{35}$$

The equation of the geodesics is then

$$\boxed{v = c_1 \int \frac{\sqrt{1 + [g'(u)]^2}\; du}{g(u)\sqrt{[g(u)]^2 - {c_1}^2}}} \tag{36}$$

The u-parameter curves are generating curves called meridians and the v - parameter curves are circles, called parallels.



We state here a theorem without proof

Theorem 1. *For a surface of revolution having parametrization* $x(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$, *any meridian is a geodesic and a parallel is a geodesic if and only if* $f'(v_0) = 0$.

Computing Geodesics of symmetric surfaces

Let us now use the derived symmetry to compute a few geodesics. We will be referring to similar surfaces later in this paper

Hyperboloid

A hyperboloid is parameterized by

$$\sigma(u, v) = (a\sqrt{1+v^2} \cdot \cos(u), a\sqrt{1+v^2} \cdot \sin(v), bv)$$

With Differential parameters

$$P = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 = a^2(1+v^2) \tag{37}$$

$$Q = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = 0 \tag{38}$$

$$R = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 = \frac{a^2v^2}{1+v^2} + b^2 \tag{39}$$

The Geodesic equation is given by substituting in equation 31 :

$$u(v) = \pm c\int_{v_0}^{v} \frac{c\sqrt{\frac{a^2v^2}{1+v^2} + b^2}}{{}^2(1+v^2)\sqrt{a^2(1+v^2) - c^2}} \tag{40}$$

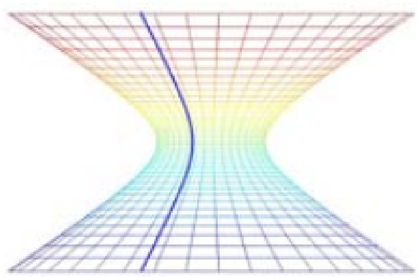
Clairaut's relation of Hyperboloid is

$$a\sqrt{(1+v^2)}\cos\theta = 0 \tag{41}$$

We consider 3 cases

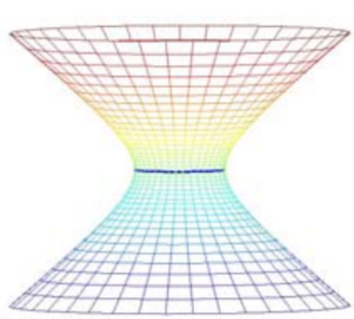
1. Meridians on the Hyperboloid are geodesics which satisfies c = 0

   Let $\alpha(s) = \sigma(u(s), v(s))$ be a meridian on the Hyperboloid. Since θ is the angle from $\sigma(u)$ to $\dot{\alpha}$, then $\theta = \pi/2$. By putting the meridian $\theta = \pi/2$ in Clairaut's relation, we have $a\sqrt{(1+v^2)} \cdot \cos(\pi/2) = c$.
   By Theorem 1 any meridian is a geodesic, hence meridians on the Hyperboloid are geodesic



2. Parallel on Hyperboloid at $v_0 = 0$ is a geodesic which satisfies c = a.

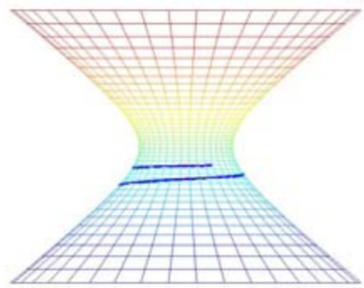
   Let $\alpha(s) = \sigma(u(s), v(s))$ be a parallel on the Hyperboloid. Since θ is the angle from $\sigma(u)$ to $\dot{\alpha}$, then $\theta = 0$. We obtain the parallel $\theta = 0$ in Clairaut's relation, we have $c = a\sqrt{(1+v^2)}$ Moreover, (Theorem 1) a parallel is a geodesic iff $f'(v_0) = 0$. Since $f(v) = a\sqrt{1+v^2}$ then $f'(v) = \frac{2av}{\sqrt{1+v^2}}$ Thus $f'(v_0) = 0$ where$v_0 = 0$. Therefore, the parallel on Hyperboloid at is geodesic which satisfies c = a



3. Other geodesics satisfy

$$|c| < |a|$$

In this case, the geodesic is not perpendicular to any meridians which satisfy $a\sqrt{(1+v_2)}\cos\theta = c$. This implies that $|c| < |a|$. For the example below, the starting point (u, v) and the direction (du/ds, dv/ds) are given.



Paraboloid

A paraboloid is parameterized by $\sigma(u,v) = (av\cdot\cos(u), av\cdot\sin(v), v^2)$

With Differential parameters

$$P = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 = a^2v^2 \tag{42}$$

$$Q = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = 0 \tag{43}$$

$$R = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 = a^2 + 4v^2 \tag{44}$$

The Geodesic equation is given by substituting in equation 31 :

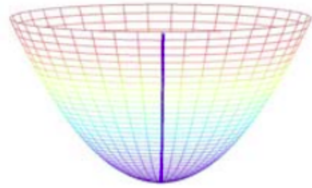
$$u(v) = \pm c\int_{v_0}^{v} \frac{c\sqrt{a^2+4v^2}}{av\sqrt{a^2v^2-c^2}} \tag{45}$$

Clairaut's relation of Paraboloid is

$$av\cos\theta = 0 \tag{46}$$

We consider 3 cases

1. Meridians on the Paraboloid are geodesics which satisfies c = 0

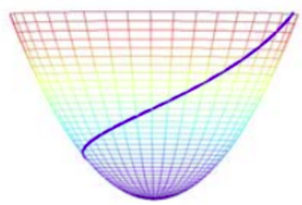


2. Parallel on Paraboloid at $v_0 = 0$ is not geodesic which satisfies c = a since $f'(v_0) = a \neq 0$ (Theorem 1).

3. . Other geodesics satisfy

$$|c| < |a|$$

.



Funnel

The Funnel is a surface of revolution obtained by rotating the curve $\ln v$.

A funnel is parameterized by $(av\cos(u), av\sin(u), \ln v)$

With Differential parameters

$$P = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 = a^2v^2 \tag{47}$$

$$Q = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = 0 \tag{48}$$

$$R = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 = a^2 + \frac{1}{v^2} \tag{49}$$

The Geodesic equation is given by substituting in equation 31 :

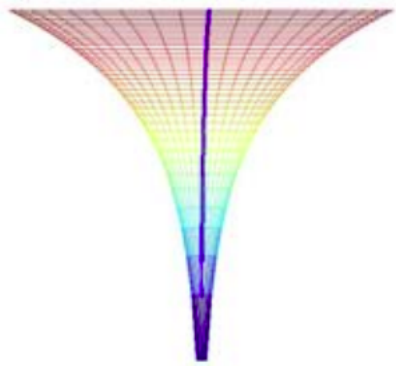
$$u(v) = \pm c\int_{v_0}^{v} \frac{c\sqrt{a^2v^2+1}}{av^2\sqrt{a^2v^2-c^2}} \tag{50}$$

Clairaut's relation of Paraboloid is

$$av \cos\theta = 0 \tag{51}$$

We consider 3 cases

1. Meridians on the Funnel are geodesics which satisfies c = 0

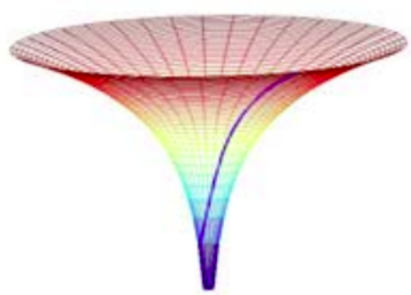


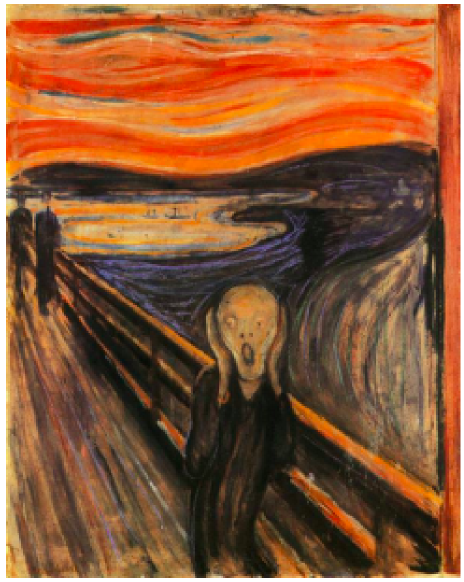
2. Parallel on Funnel at $v_0 = 0$ is not geodesic which satisfies c = a since $f'(v_0) = a \neq 0$ (Theorem 1).

3. . Other geodesics satisfy

$$|c| < |a|$$

.





https://www.edvardmunch.org/the-scream.jsp

Figure 1: The Confused Screaming: The reaction while trying to understand this maths behind all the physics.

The Spacetime Curvature

Space is flat........

We can now use all these concepts to explain the phenomenon of flat or curved space. Consider three points in space, and join them with the light from laser beams, if you find that the triangle doesn't have the expected sum of angles of 180°, that means that space is curved. And when the angles do add up to 180°, that is what it means for space to be flat. Mass also affects the overall geometry of the universe. The energy and the density of matter in the universe determines whether the universe is flat, open or closed.

If the density is equal to the critical density, then the universe has zero curvature; it is flat. You can imagine this flat universe to be like a sheet of paper that extends infinitely in all directions. A universe with density greater than the critical density makes a closed universe. This can be imagined as the surface of a sphere and it has positive curvature. And if the universe's density is less than the critical density, then the universe is open and has negative curvature, like the surface of a saddle or a pringle.

Measurements from the Wilkinson Microwave Anisotropy Probe (WMAP) have shown the observable universe to have a density very close to the critical density (within a 0.4% margin of error). Of course, the observable universe may be many orders of magnitude smaller than the whole universe. But the part of the universe we can observe appears to be fairly flat.

.......But the Spacetime Is Not!

Now, the space is flat, but space-time isn't. How do we know that? According to Einstein's theory of general relativity, massive objects warp the spacetime around them. In classical physics, time proceeds constantly and independently for all objects. In relativity, spacetime is a four-dimensional continuum combining the familiar three dimensions of space with the dimension of time.

This can be explained with the help of the analogy by considering space-time as a rubber sheet that can be deformed. In any region distant from massive cosmic objects such as stars, space-time is uncurved—that is, the rubber sheet is absolutely flat. If one were to probe space-time in that region by sending out a ray of light or a test body, both the ray and the body would travel in perfectly straight lines, like a child's marble rolling across the rubber sheet.

However, the presence of a massive body curves space-time, as if a bowling ball were placed on the rubber sheet to create a cuplike depression. In the analogy, a marble placed near the depression rolls down the slope toward the bowling ball as if pulled by a force. Besides, if the marble is given a sideways push, it will describe an orbit around the bowling ball, as if a steady pull toward the ball is swinging the marble into a closed path.

And this is how Einstein's theory explains the gravitational force to be. According to Einstein's general theory of relativity, gravity is no longer a force that acts on massive bodies, as viewed by Newton's universal gravitation. Instead, general relativity links gravity to the geometry of spacetime itself, and particularly to its curvature. Gravity, in Einstein's terms, can be considered to be the effect the warping of spacetime around the massive bodies has on other objects. Thus, to account for gravity in relativity, the structure of the four-dimensional spacetime is not 'flat' but is curved by the presence of massive bodies.

In the following figure, artistic representation visualizes spacetime as a simplified, two-dimensional surface, which is being distorted by the presence of three massive bodies, represented as colored spheres. The distortion caused by each sphere is proportional to its mass. Now, a point to note here is that space-time is curved, but space itself is flat. And almost all the visual representations of "space-time" curvature depict the spacial curvature only. This is so because minds of the most of us (including mine) are not capable of imagining the fourth dimension, imagining the time as the fourth dimension

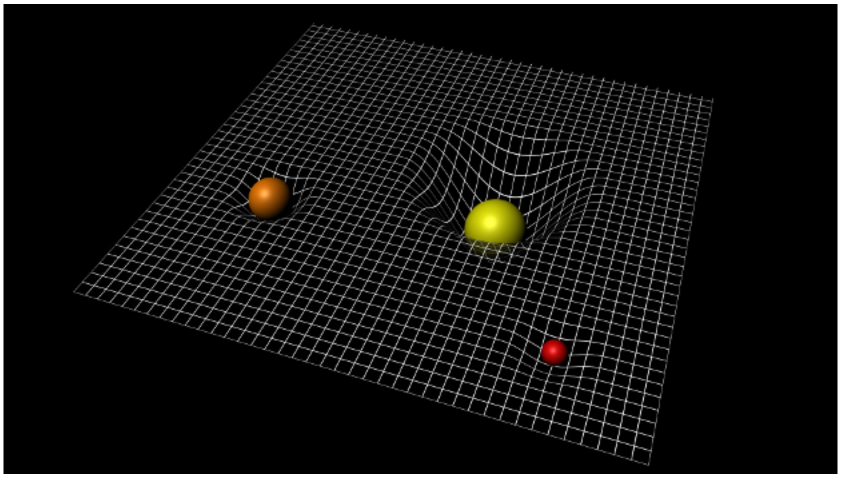


Figure 2: Two dimensional surface being distorted by the presence of three massive bodies with distortion proportional to their masses.

is far away. But while using these pictures as an aid for our imagination, we should keep in mind that they are not what we will call the precise depictions of space-time curvature.

Home Project:

To explore the above idea of rubber sheet analogy, we tried to make a toy model to show how spacetime curvature actually works. We used a bedsheet instead of the rubber sheet in the above analogy (since none of us had a rubber sheet) representing the spacetime (as we've mentioned before, this gives a false sense of actual spacetime as spacetime is a set of events and not of locations, and this model gives the sense of spacial curvature only. But making an accurate of model of something which we can barely imagine is difficult. So, we'll work with this.).

The figure (a) represents the uncurved or flat spacetime without any mass. If the red lines on the bedsheet are considered to be the path of some object, then as we can see, it's perfectly straight.

The figure (b) shows the bedsheet with a heavy object (mortar) placed at the center representing curved spacetime due to the mass. The two red lines on the either side of the mass are clearly bent inwards due to the depression created by the mass. This shows how the massive bodies warp spacetime.

For the figure (c), we dropped a ball at the edge of the bedsheet to observe the path of object near the mass in spacetime. It didn't come out to be perfectly spiral as was expected, probably because the ball was too big. But near the mortar, it does spiral around it. This path represents the geodesic.

(a) Bedsheet representing the uncurved or flat spacetime without any mass.

(b) Bedsheet with a mortar placed at the center showing how the massive bodies warp spacetime.

(c) The path of the ball around the mortar crudely representing the geodesic.

Figure 3: A toy model explaining spacetime curvature and the path traced by an object near a massive body.

The Geodesics:

Now, where in this all sci-fi physics do geodesics fall? As we saw above, gravity is a warping of spacetime. Stars like our sun, which have strong gravitational force, actually bend and stretch the fabric of the universe itself. But how does this theory explain the day-to-day phenomenon we observe, like the falling of two apples towards each other while free-falling under the earth's gravitational force? Well, according to Einstein, the warping of spacetime makes the objects to travel on curved paths near massive objects (this can be picturized using the marble analogy). That simply means that the phenomenon of attracting apples which the flat spacetime explained assuming that they fall radially along straight lines towards the center of the earth due to gravitational force, can be explained using curved spacetime by assuming that instead of straight lines, the apples actually travel along straight lines on the curved surface.

This changes our notion of what really a straight line in curved spacetime is. Instead of calling them straight lines, we call them geodesics, which represent the straightest possible path of an object in curved spacetime. It is the smallest curve joining any two points on the surface. It can also be defined as a self-parallel curve, i.e., a curve whose tangent vector $\mathbf{t}$ satisfies $\nabla_t \mathbf{t} = \kappa t$, where κ is any scalar function.

The mathematical significance of the geodesic, in the way that we understood it, is explained above. The final equation of geodesic is the equation of motion of freely-falling particles in a curved space-time. Particles traveling on a geodesic feel no forces. It is a curved spacetime analogue of straight lines in flat spacetime. As John A. Wheeler famously wrote general relativity can be summarised in one sentence: "Space-time tells matter how to move; matter tells space-time how to curve".

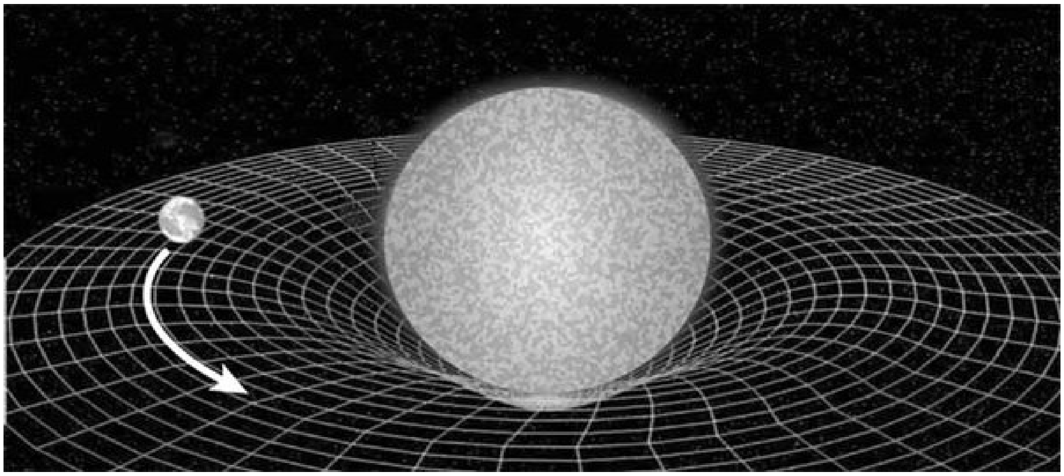


Figure 4: Effect of warping of spacetime by a massive body on other objects.

The above image also explains the path of planetary orbits which are circles expressed as geodesics of the sphere.

Time Dilation

Gravitational fields are really a bending of space and time, the geodesics through those regions of space are also bent. In space, light is bent in gravitational fields and travels along geodesics.

Time goes faster the farther you are from the earth's surface compared to the time on the surface of the earth. This effect is known as "gravitational time dilation". Gravitational time dilation occurs because objects with a lot of mass create a strong gravitational field. The stronger the gravity, the more spacetime curves, and slower the time proceeds.

According to the theory of general relativity, time passes at different rates at different levels of gravitational potential. Time passes slower in regions of high gravitational potential than it does in regions of low gravitational potential. This theory predicts that clock at different altitudes would run at different rates.

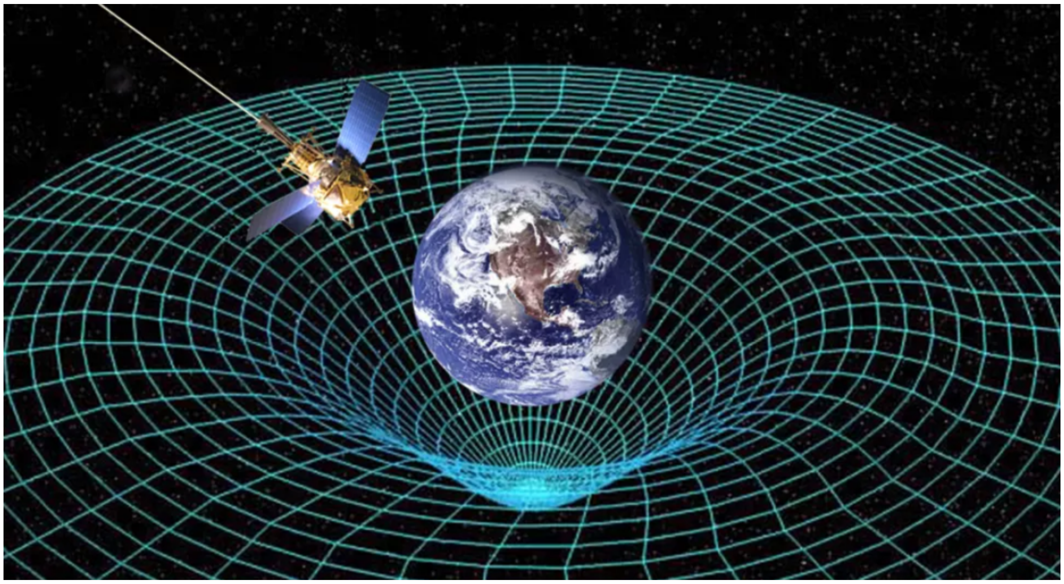


Figure 5: Earth's mass warps space and time so that clocks in a "gravity well" (a pull of gravity that a large body in space exerts) run slower and run faster when outside the well (this is really time dilation due to spacetime curvature). Although this is a very weak effect, the time difference can be measured on the scale of meters using atomic clocks.

Black Holes:

Talking about the most convoluted curvatures of spacetime, we come to black-holes. They warp the spacetime such that even light will travel on the highly curved paths. Photons travel on the geodesic paths in spacetime. Near the black hole, the curvature becomes so high that the light bends into the path that all terminates at the blackhole's singularity.

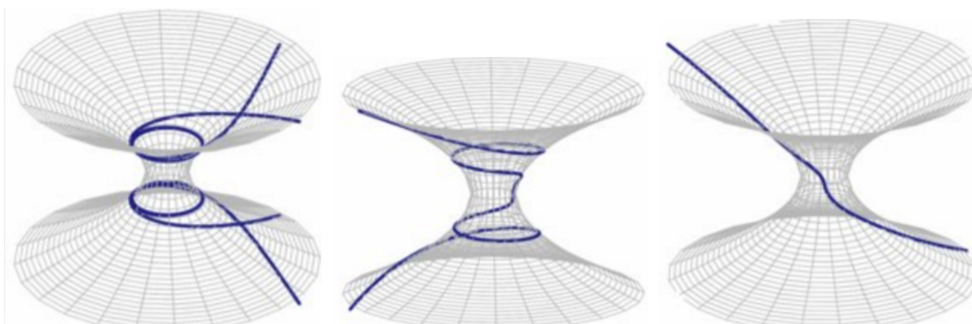
In 1916 the German astronomer Karl Schwarzschild described a new effect. If the mass is concentrated in a vanishingly small volume, a singularity, gravity will become so strong that nothing pulled into the surrounding region can ever leave. Even light cannot escape. In the rubber sheet analogy, it as if a tiny massive object creates a depression so steep that nothing can escape it. In recognition that this severe space-time distortion would be invisible—because it would absorb light and never emit any—it was dubbed a black hole.

In quantitative terms, Schwarzschild's result defines a sphere that is centered at the singularity and whose radius depends on the density of the enclosed mass. Events within the sphere are forever isolated from the remainder of the universe; for this reason, the Schwarzschild radius is called the event horizon.

Wormholes

In general relativity, a wormhole is considered to be a tunnel through which two distant regions of spacetime can be connected. Wormholes contain two mouths, with a throat connecting the two.

Ellis wormhole is a nongravitating, purely geometric, traversable wormhole. Since there is no gravity in force, an inertial observer (test particle - can be a photon) can sit forever at rest at any point in space, but if set in motion will follow a geodesic of an equatorial cross section at constant speed.



(a) Geodesics confined to one side of the wormhole throat

(b) Geodesics spiraling onto the wormhole throat throughout

(c) Geodesics passing through the wormhole throat

Figure 6: Three examples of geodesics though a wormhole

Morris and Thorne used Ellis wormhole as a tool for teaching general relativity.

Morris and Thorne wormhole metric:

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

Geodesic Equations:

$$\frac{d^2t}{d\tau^2} = 0$$

$$\frac{d^2r}{d\tau^2} = r\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]$$

$$\frac{d}{d\tau}\left((b^2 + r^2)\frac{d\theta}{d\tau}\right) = (b^2 + r^2)\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2$$

$$\frac{d}{d\tau}\left((b^2 + r^2)\sin^2\theta\frac{d\phi}{d\tau}\right) = 0$$

Other applications of geodesics

1. **Geodesic Domes and airframes -** Geodesic domes enclose large volume of space using very less amount of construction material, are extremely lightweight and have great strength. They have also withstood hurricanes, earthquakes, and fires better than rectangle-based structures. They've been used for military radar systems, churches, auditoriums and planetariums.

   Geodesic airframes make use of a space frame and are like a basket-weave of load bearing members. They are comparatively lighter and stronger and since they are spaceframes with only the outer part having the geodetic structure, the centre has a large empty space that can be used to take payload or fuel.

2. **Geodesy -** Geodesy is the science of accurately measuring and understanding three properties of the Earth: its geometric shape, its orientation in space, and its gravitational field. One of the main purposes of geodesy is to establish a reference system, define a set of points (known as geodesic vertices) which form a geodesic network and based on these points, coordinates for any point on the Earth's surface can be computed. Global Positioning System (GPS) is a space-based tool used by geodesists to measure such points on the Earth's surface.

3. **Geodesic Methods in Computer Vision and Graphics -** It includes several applications of the numerical computation of geodesic distances and shortest paths to problems in surface and shape processing, in image segmentation, motion planning, sampling, meshing, comparison of shapes and UV mapping.

4. **Medical image analysis and image segmentation -** Geodesic deformable models are used for medical image analysis. Semiautomatic segmentation method based on the geodesic distance transform, addresses a significant need in the field of neuro-oncology to obtain accurate tumor volumes without the need for manual segmentation.

References

[1] Astonomy Lecture Number 21,
`https://web.njit.edu/ gary/202/Lecture21.html`

[2] Does time go faster at the top of a buiding compared to the bottom?
`https://wtamu.edu/ cbaird/sq/2013/06/24/does-time-go-faster-at-the-top-of-a-building-compared-to-the-bottom/`

[3] Geodesic Equations for the Wormhole Metric,
https://people.uncw.edu/hermanr/GRcosmo/GeodesicGR.pdf

[4] Ellis Wormhole
https://en.wikipedia.org/wiki/Elliswormhole

[5] Origin and Uses of Geodesic Domes,
https://science.howstuffworks.com/engineering/structural/geodesic-dome1.html

[6] Geodetic Aircraft Design,
https://www.barneswallisfoundation.co.uk/life-and-work/geodetic-aircraft-design/

[7] Fundamentals of cartography and geodesy,
https://volaya.github.io/gis-book/en/Cartography.html

[8] Geodesic Methods in Computer Vision and Graphics,
https://hal.archives-ouvertes.fr/hal-00528999/document

[9] Geodesic deformable models for medical image analysis,
https://ieeexplore.ieee.org/document/730407

[10] Automated Tumor Volumetry Using Computer-Aided Image Segmentation,
https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4395536/

[11] Wongvisarut Khuangsatung and Pakkinee Chitsakul; "Some Geodesics in Open Surfaces Classified by Clairaut's Relation"
https://zenodo.org/record/1070305/files/8123.pdf

[12] Rowland, Todd and Weisstein, Eric W. "Geodesic." From MathWorld–A Wolfram Web Resource.
https://mathworld.wolfram.com/Geodesic.html

[13] What Do You Mean, the Universe Is Flat? -Blog Article
https://blogs.scientificamerican.com/degrees-of-freedom/httpblogsscientificamericancomdegrees-of-freedom20110731what-do-you-mean-the-universe-is-flat-part-ii/

[14] IS SPACE FLAT OR CURVED?- Astronomy Resources
https://skyandtelescope.org/astronomy-resources/curvature-of-spacece/: :text=According%20to%20Einstein's%20theory%20of,around%20every%20object%20with%20mass

[15] Curved space time and geometric gravitation
https://www.britannica.com/science/relativity/Curved-space-time-and-geometric-gravitation

[16] SPACETIME CURVATURE- Image source (ESA Site)
https://sci.esa.int/web/lisa-pathfinder/-/56434-spacetime-curvature : :text=The%20curvature%20of %20spacetime%20influences,interaction %20between%20matter%20and%20spacetime

[17] Lectures given at the African Institute for Mathematical Sciences, Cameroon (AIMS-Cameroon), in January 2018 and January 2019 "Gravitation:From Newton to Einstein".
Author: Pierre Fleury Département de Physique Théorique Université de Genève, Switzerland

[18] General Relativity PHY-5-GenRel U01429 16 lectures
Author: Alan Heavens, School of Physics, University of Edinburgh Département de Physique Théorique Université de Genève, Switzerland

[19] http://astronomy.swin.edu.au/ cblake/Class7:Geodesics.pdf

[20] Euclidean and Non-Euclidean Geometry
https://www.cs.unm.edu/ joel/NonEuclid/noneuclidean.html

[21] The front page
https://github.com/dev-aditya/latex_template/tree/main/Notes